The Equal Area, Log-Linear and Log-Tchebycheff Rules: Derivations and Assumptions

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ABSTRACT

The volumetric flow rate in a duct is commonly calculated based on the arithmetic average of an array of point velocity measurements across the flow. Various traverse patterns, such as the Equal Area, Log-Linear and Log-Tchebycheff rules, are defined for round and rectangular ducts in ASHRAE and ISO standards. While many studies on the accuracy of these rules have been published, there is little discussion on the reasons for this accuracy, i.e., the assumptions (if any) about the underlying velocity profile. Such knowledge would be useful to engineers involved in ventilation system testing and commissioning, particularly where some compromise is necessary, e.g., for non-ideal traverse locations or duct cross-sections. This paper explains how the three rules are derived, reveals their underlying assumptions, and suggests some ideas for further research.

INTRODUCTION

One method for determining the volumetric flow rate in a conduit is to measure the streamwise velocity at an array of points across the flow and multiply their average by the cross-sectional area. Three standard rules for determining the measurement point coordinates are recommended in current ASHRAE and ISO standards – the Equal Area, Log-Linear and Log-Tchebycheff rules (e.g., ISO 2008, ISO 2020, ASHRAE 2021). These rules have been shown to be acceptably accurate in many practical situations (Winternitz and Fischl 1957, Kinghorn and McHugh 1977, Hickman 2015). However, a possible weakness in the standards is the lack of explanation for why these rules work so well, and why they are recommended for some situations and not others. For example:

1. Why does ASHRAE 111 (ASHRAE, 2017) define the Equal Area rule for a rectangular duct but not for a circular duct?
2. Why do the ASHRAE and ISO standards present an (unweighted) log-linear rule for round ducts but not for rectangular ducts?
3. Why are the Log-Linear and Log-T coordinates defined with such precision?
4. To what extent can adjustments be tolerated (to individual points or to the traverse location)?
5. Which rule is most economical (i.e., most accurate for a given number of measurements) for a particular duct geometry and flow condition?

A first step towards addressing such questions, and the subject of this paper, is to determine how the coordinates

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for each rule are calculated. Thus, the underlying assumptions about the velocity profile (if any) are revealed, some implications for flow measurement are highlighted, and a foundation is laid for further study.

EQUAL AREA RULE

Round Duct Traverse

In the Equal Area rule for a round duct, the cross-section is subdivided into multiple rings and a central circle, all of equal area (Ower & Pankhurst 1977). Velocity is measured at concentric circles which bisect each ring into two sub-rings of equal area. While no assumption is made regarding the velocity profile, the measurement resolution increases towards the wall. Hence this method may be particularly effective for fully developed flows with a large boundary layer region (i.e., laminar or low Reynolds number).

Rectangular Duct Traverse

In a traverse across a rectangular duct, the duct width or height is subdivided into intervals of equal width, and velocity is measured at the midpoint of each interval (ASHRAE 2017). Again, no assumption is made regarding the velocity profile.

Conflicting conclusions regarding the accuracy of this method (relative to the Log-Tchebycheff rule) have been reported in the literature (e.g., McFerran 1999, Klaassen & House 2001, Zhou 2005, Hickman 2015). The insight into the Log-T rule provided in this paper will enable comparison on a more fundamental, mathematical basis in future work.

LOG-LINEAR RULE

Round Duct Traverse

The Log-Linear rule was originally derived by Winternitz & Fischl (1957) for a round duct only. It is based on the logarithmic law of the wall, but with an additional linear term (which influences the slope of the velocity profile away from the wall). The following derivation is simpler than the original approach due to the choice of coordinate system.

Traverse With an Even Number of Points. For an even number of points, the pipe is subdivided into $n = j/2$ concentric regions of equal area, as shown in Figure 1a. Within each region m, two velocity measurement points ρ_{m1} and ρ_{m2} are identified such that their area-based average is equal to the area-based average of the log-linear function,

$$f(\rho) = C_1 + C_2 \ln(1-\rho) + C_3(1-\rho) \tag{1}$$

where $\rho = r/R$ is the non-dimensional radial position (measured from the axis), R is the pipe radius and C_1, C_2 and C_3 are arbitrary coefficients. By choosing convenient values for these coefficients, a set of simultaneous equations for ρ_{m1} and ρ_{m2} can be constructed, as shown in Equation 2. The resulting values of ρ_{m1} and ρ_{m2} then remain valid for *any* values of C_1, C_2 and C_3.

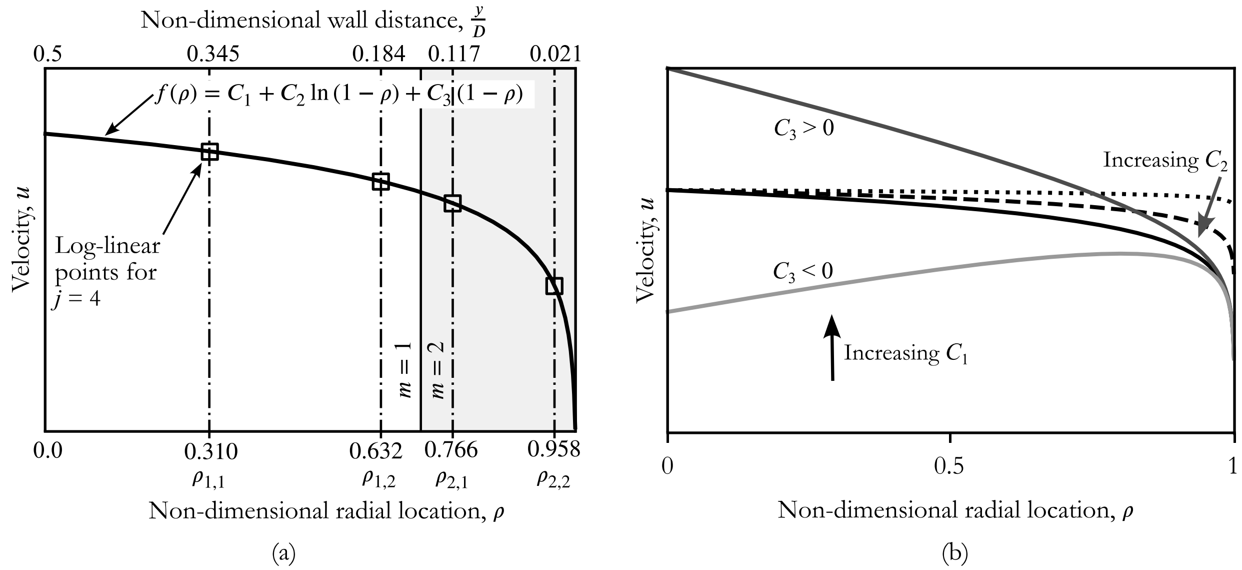

Figure 1 (a) Illustration of the Log-Linear rule applied to a round duct, with an even number of points per radius ($j = 4$), as described in Equation 7. (b) Illustration of the effects of the coefficients C_1, C_2 and C_3 on Equation 1.

$$C_1 = C_3 = 0; C_2 = 1: \quad \int_{\sqrt{\frac{m-1}{n}}}^{\sqrt{\frac{m}{n}}} \ln(1-\rho)\, 2\pi\rho \, \mathrm{d}\rho = \frac{1}{2}[\ln(1-\rho_{m1}) + \ln(1-\rho_{m2})] \cdot \frac{\pi\, 1^2}{n} \tag{2a}$$

$$C_1 = C_2 = 0; C_3 = 1: \quad \int_{\sqrt{\frac{m-1}{n}}}^{\sqrt{\frac{m}{n}}} (1-\rho)\, 2\pi\rho \, \mathrm{d}\rho = \frac{1}{2}[(1-\rho_{m1}) + (1-\rho_{m2})] \cdot \frac{\pi\, 1^2}{n} \tag{2b}$$

Equations 2a and 2b are solved following the approach of Winternitz and Fischl (1957), where the intermediate variables a_m and b_m are introduced to define the following set of simultaneous equations:

$$\frac{1}{2}[\ln(1-\rho_{m1}) + \ln(1-\rho_{m2})] = \ln b_m \tag{3a}$$

$$\frac{1}{2}[(1-\rho_{m1}) + (1-\rho_{m2})] = a_m \tag{3b}$$

These can be rearranged as

$$(1-\rho_{m1})(1-\rho_{m2}) = b_m^2 \tag{4a}$$

$$(1-\rho_{m1}) + (1-\rho_{m2}) = 2a_m \tag{4b}$$

which are recognizable as the product and sum of roots of the quadratic equation $(1-\rho)^2 - 2a_m(1-\rho) + b_m^2 = 0$. Thus, the standard solution for a quadratic equation can be used to obtain separate expressions for ρ_{m1} and ρ_{m2} (note the $\pm$ sign):

$$\rho_m = 1 - \left(a_m \pm \sqrt{a_m^2 - b_m^2}\right) \tag{5}$$

Separately, the values of a_m and b_m can be calculated by combining Equations 2 and 3 (note that π cancels out):

$$\ln b_m = n \int_{\sqrt{\frac{m-1}{n}}}^{\sqrt{\frac{m}{n}}} \ln(1-\rho)\, 2\rho \, \mathrm{d}\rho \tag{6a}$$

$$a_m = n \int_{\sqrt{\frac{m-1}{n}}}^{\sqrt{\frac{m}{n}}} (1-\rho)\, 2\rho \,\mathrm{d}\rho \tag{6b}$$

and substituting the results into Equation 5. There are two solutions to Equation 5 (note the $\pm$ sign):

$$\rho_m = 1 - n\left(\frac{2\rho_1^3}{3} - \rho_1^2 - \frac{2\rho_2^3}{3} + \rho_2^2\right) \pm \left\{ n^2 \left(\frac{2\rho_1^3}{3} - \rho_1^2 - \frac{2\rho_2^3}{3} + \rho_2^2\right)^2 - \exp\left[2n\left(-\rho_1^2 \ln(1-\rho_1) + \frac{\rho_1^2}{2} + \rho_1 + \rho_2^2 \ln(1-\rho_2) - \frac{\rho_2^2}{2} - \rho_2 + \ln(|\rho_1 - 1|) - \ln(|\rho_2 - 1|)\right)\right]\right\}^{\frac{1}{2}} \tag{7}$$

where $\rho_1 = \sqrt{(m-1)/n}$ and $\rho_2 = \sqrt{m/n}$ are the boundaries of the mth interval, i.e. the limits of integration in Equation 2. In the outer ring, where $m = n$, Equation 2a becomes an imperfect integral with a vertical asymptote at the wall, and hence the upper limit becomes $\rho \to 1$. To solve Equation 7 in the outer ring, the upper limit can be set to $\rho_2 = 1 - \epsilon$, where ϵ is a very small number (e.g., 10^{-6})*.

The full series of radial measurement points can be expressed in terms of wall distance non-dimensionalized by duct diameter using the transformation $y_i/D = (1-\rho_i)/2$.

Traverse With an Odd Number of Points. If the number of points per radius j is odd, the cross-section is subdivided into $n = (j+1)/2$ concentric regions. The outer, wall-adjacent region has half the area of the other regions ($A_{\text{outer}} = A/j$), and velocity is measured at the point, ρ_{outer}, where it equals the mean of a logarithmic velocity profile, $f(\rho) = C_1 + C_2 \ln(1-\rho)$. Setting $C_1 = 0$ and $C_2 = 1$ yields the equation

$$\int_{\sqrt{1-\frac{1}{j}}}^{1} \ln(1-\rho)\, 2\pi\rho \,\mathrm{d}\rho = \ln(1-\rho_{\text{outer}}) \cdot \frac{\pi}{j} \tag{8}$$

The solution to Equation 8 is

$$\rho_{\text{outer}} = 1 - \exp\left[j\sqrt{1-\frac{1}{j}} - \left(1-\frac{1}{j}\right)\ln\left(1-\sqrt{1-\frac{1}{j}}\right) + \ln\left(\left|\sqrt{1-\frac{1}{j}} - 1\right|\right) - 1 - \frac{1}{2j}\right] \tag{9}$$

While the coefficients in Equation 8 have been set to $C_1 = 0$ and $C_2 = 1$, the resulting value of ρ_{outer} remains valid for all values of C_1 and C_2.

For the core region, Equation 2 is used, but the integration limits on the left-hand side are set to $\rho_1 = \sqrt{2(m-1)/(2n-1)}$ and $\rho_2 = \sqrt{2m/(2n-1)}$, and on the right-hand side the area of each ring is changed to $2\pi\, 1^2/(2n-1)$. The points ρ_{m1} and ρ_{m2} are then calculated as (note the $\pm$ sign):

$$\rho_m = 1 - \left(n-\frac{1}{2}\right)\left(\frac{2\rho_1^3}{3} - \rho_1^2 - \frac{2\rho_2^3}{3} + \rho_2^2\right) \pm \left\{ \left(n-\frac{1}{2}\right)^2 \left(\frac{2\rho_1^3}{3} - \rho_1^2 - \frac{2\rho_2^3}{3} + \rho_2^2\right)^2 - \exp\left[2\left(n-\frac{1}{2}\right)\left(-\rho_1^2 \ln(1-\rho_1) + \frac{\rho_1^2}{2} + \rho_1 + \rho_2^2 \ln(1-\rho_2) - \frac{\rho_2^2}{2} - \rho_2 + \ln(|\rho_1 - 1|) - \ln(|\rho_2 - 1|)\right)\right]\right\}^{\frac{1}{2}} \tag{10}$$

* Note that when solving Equations 7 and 10 in the spreadsheet software Microsoft Excel, brackets must be used to override the default order of operations, where negation (-) incorrectly takes precedence over exponentiation (^).

Rectangular Duct Traverse

A log-linear rule can be derived for a traverse across a rectangular duct by recasting Equation 1 in terms of the duct half-width, S,

$$f\left(\frac{x}{S}\right) = C_1 + C_2 \ln\left(\frac{x}{S}\right) + C_3\left(\frac{x}{S}\right) \tag{1b}$$

and following the procedure outlined above (with the integrals in Equations 2, 6 and 8 adjusted appropriately). This method would be more general than the rectangular log-linear rule defined in ISO 3966 (ISO 2020), which relies on a weighted average of exactly 26 point measurements (Miles et al 1966, Ower and Pankhurst 1977), and thus may merit further investigation.

Discussion

The Log-Linear rule should be accurate for any velocity profile which has the form of Equation 1. The coefficients C_1, C_2 and C_3 can have any value, and their effects on the velocity profile are illustrated in Figure 1b. The constant term C_1 shifts the whole profile vertically, the coefficient C_2 varies the curvature of the profile at the wall and the coefficient C_3 influences the slope in the core region. Furthermore, these coefficients need not be constant for all n intervals across the profile, i.e., the assumed profile is *piecewise* log-linear. For example, Winternitz & Fischl (1957) observed very good agreement for a range of asymmetric velocity profiles.

LOG-TCHEBYCHEFF RULE

In the Log-Tchebycheff rule, the duct cross-section is subdivided into an outer region, where the velocity is represented by a logarithmic function, and a core region, where a polynomial function is assumed. The measurement locations are distributed unequally (as described below) such that their arithmetic average equals the average of this assumed profile along the traverse path.

Rectangular Duct Traverse

Figure 2 illustrates a Log-T traverse across a rectangular duct with $j = 5$ points. In each outer region, a single measurement point X_{outer} is located such that its value equals the mean of the logarithmic function $f(X) = C_1 + C_2 \ln(X)$, where $X = x/W$, W is channel width and the coefficients C_1 and C_2 can take *any* value. This point is calculated by setting $C_1 = 0$, $C_2 = 1$, and solving the equation

$$\ln(X_{\mathrm{outer}})\frac{1}{j} = \int_0^{\frac{1}{j}} \ln(X)\,\mathrm{d}X \tag{11}$$

where the non-dimensional width of the outer region is $1/j$. Thus, one outer measurement point is at

$$X_{\mathrm{outer},1} = j\exp\left[\frac{\ln\left(\frac{1}{j}\right) - 1}{j}\right] \tag{12}$$

and the other outer point is $X_{\mathrm{outer},2} = 1 - X_{\mathrm{outer},1}$. These points remain valid for *any* values of C_1 and C_2.

In the core region, Chebyshev* quadrature is applied to calculate the average value of the assumed polynomial velocity profile. Chebyshev quadrature is a method of numerically integrating a polynomial function, $f(x) = \sum_{i=1}^{p} C_i x^i$, based on its values at a small number of points at particular locations, x_{Cheb}. The standard formula, defined for the

* There are a variety of transliterations of the original Russian name from Cyrillic script to Latin script.

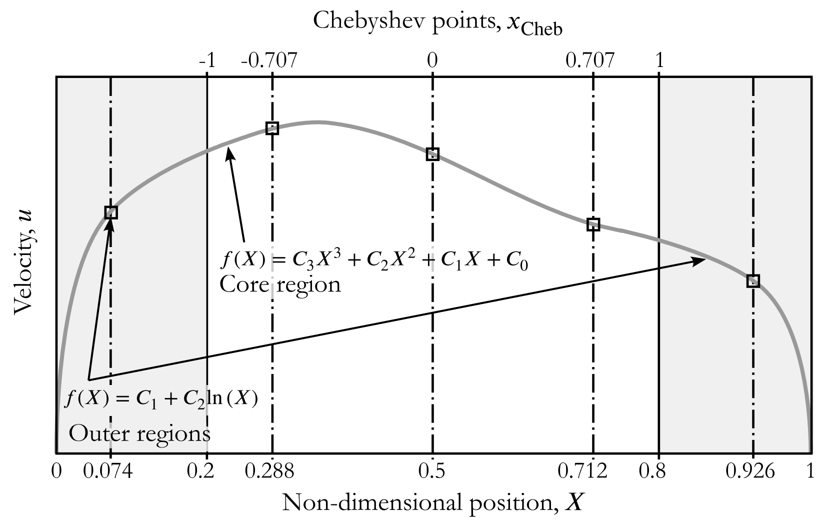

Figure 2 Illustration of the Log-T rule for a five-point traverse of a rectangular duct, as described in Equations 12 and 14. The square markers indicate the measured point velocities and the solid grey line is the assumed velocity profile.

interval $-1 < x < 1$, is

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \frac{2}{n}\sum_{i=1}^{k} f\big(x_{\text{Cheb},i}\big) + E \tag{13}$$

where E is the error term (Chebyshev 1874, Hildebrand 1956). The method is *exact* (i.e., the error term E is zero) for polynomials of degree k when k is odd, and of degree $k + 1$ when k is even. The series of points $x_{\text{Cheb},i}$ are listed for $2 \le k \le 6$ in Table 1. Proofs of the accuracy of Equation 13 and methods for computing the points in Table 1 have been published by Hildebrand (1956) and Gautschi (1976).

The number of core measurement points for a traverse across a rectangular duct is $k = j - 2$. The equation to transform the corresponding points in Table 1 from the standard interval $-1 < x_{\text{Cheb,i}} < 1$ to the core region, $1/j < X_i < 1 - 1/j$ is

$$X_i = \big(1 + x_{\text{Cheb},i}\big)\left(\frac{1}{2} - \frac{1}{j}\right) + \frac{1}{j} \tag{14}$$

Figure 2 shows how the three core points ($k = 3$) are defined for a five-point traverse ($j = 5$). The standard interval for Chebyshev quadrature is indicated by the horizontal axis at the top of the plot, and corresponds to the core region, $1/5 < X < (1 - 1/5)$, along the horizontal axis at the bottom of the plot. Equation 14 is used to transform the Chebyshev points from $x_{\text{Cheb},i} = \{0, \pm 0.707\}$ (Table 1) to values of X of 0.288, 0.5 and 0.712. Because k is odd, the velocity profile is assumed to have the form of *any* polynomial of degree $k \le 3$.

Table 1. Points for Chebyshev Quadrature (Hildebrand 1956).

Number of Points, k	Chebyshev Points, $x_{\text{Cheb},i}$
2	±0.577350
3	0; ±0.707107
4	±0.187592; ±0.794654
5	0; ±0.374541; ±0.832497
6	±0.266635; ±0.422519; ±0.866247

Round Duct Traverse

For a round duct, the outer region is an annulus of area $A_{\text{outer}} = A/j$, where A is the total cross-sectional area and j is the number of velocity measurement points per radius. Thus, the boundary between the outer and core regions is at the non-dimensional radius $\rho_{\text{core}} = R_{\text{core}}/R = \sqrt{1 - 1/j}$, where R is the duct radius. The velocity profile in this region is assumed to be logarithmic and is measured at the radial location which corresponds to the area-weighted average of a logarithmic profile (Equations 8 and 9).

In the core region, the velocity profile is assumed to have the form of a polynomial equation, and the remaining $k = j - 1$ measurement points are determined using Chebyshev quadrature. Noting that the local annular area is proportional to radius squared, the measurement points in the core region are determined by transforming the Chebyshev quadrature points from their original interval $-1 < x_{\text{Cheb},i} < 1$ (Table 1) to the interval $0 < \rho_i^2 < \rho_{\text{core}}^2$, yielding:

$$\rho_i = \sqrt{\frac{1}{2}\left(1 - \frac{1}{j}\right)\left(1 + x_{\text{Cheb},i}\right)} \tag{15}$$

The full series of radial measurement points in the core and annular regions can be recast in terms of non-dimensional wall distance using the transformation $y_i/D = (1 - \rho_i)/2$, where D is duct diameter.

Discussion

Interesting Features of Chebyshev Quadrature. An example is presented to illustrate the rather remarkable ability of Chebyshev quadrature to accurately integrate polynomial curves of degree up to $k + 1$ with only a few points. Four somewhat arbitrary polynomial functions of increasing degree are defined in Table 2. The polynomial degree is listed in terms of k alongside integral solutions by the conventional, analytical approach and by Chebyshev quadrature for $k = 2$ points (see Figure 3). As expected, the numerical method is perfectly accurate for the first three polynomials and incorrect for the fourth case, where the polynomial degree ($p = 4$) exceeds $k + 1 = 3$. While this feature of Chebyshev quadrature may be surprising at first, it is analogous to evaluating the integral of a straight line (a polynomial of degree one) based on its value at only one point (its midpoint). It is easy to see that any straight line passing through the point (0, 2) in Figure 3 will have an integral of 4.0, regardless of its slope.

A defining characteristic of Chebyshev quadrature is that each function evaluation, $f(x_{\text{Cheb}})$, has an equal weight and thus an arithmetic average is valid. The function evaluations are actually weighted by interval width, corresponding to the unequal distribution of sampling points. One practical benefit of this feature is that any errors in velocity measurement would not be exacerbated (by weighting coefficients). This feature may also have been particularly desirable prior to the digital age, as fewer calculation steps are involved relative to weighted integration methods such as Gaussian quadrature and Simpson's Rule. It may now be worthwhile reconsidering the use of weighted integration rules as they may be more economical in practice.

Table 2. Definitions and Integrals of Polynomials of Increasing Degree

Polynomial	Degree, p	Analytical Solution	Chebyshev Quadrature ($k = 2$)
$P_1 = x + 2$	$p = k - 1 = 1$	4	4
$P_2 = -x^2 + x + 2$	$p = k = 2$	$3\frac{1}{3}$	$3.33\dot{3}$
$P_3 = -2x^3 - x^2 + x + 2$	$p = k + 1 = 3$	$3\frac{1}{3}$	$3.33\dot{3}$
$P_4 = x^4 - 2x^3 - x^2 + x + 2$	$p = k + 2 = 4$	$3\frac{11}{15} = 3.73\dot{3}$	$3.55\dot{5}$

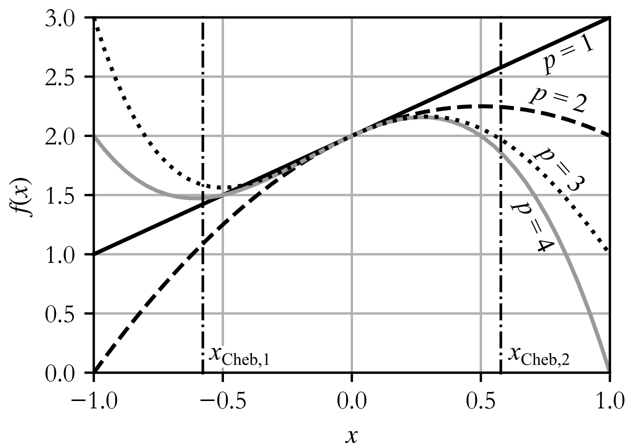

Figure 3 Plots of polynomials of increasing degree, $1 \leq p \leq 4$. The Chebyshev quadrature points, $x_{\mathsf{Cheb},i}$, for $k = 2$ points are indicated by the vertical dash-dot lines.

Assumed Velocity Profile. The derivations above show that the velocity profile assumed by the Log-Tchebycheff rule features logarithmic regions at the walls and a polynomial region in the core flow. The velocity profile in the core flow can take the form of *any* polynomial equation up to degree k (for an odd number of core points) or $k + 1$ (for an even number of core points). Therefore, the Log-Tchebycheff rule can be expected to perform quite well for a broad range of smooth velocity profiles, and it is likely to be more broadly applicable than the Log-Linear rule (in round and rectangular ducts). The Log-Tchebycheff rule would not work so well for profiles with a sudden velocity change, for example in regions with flow separation or recirculation, but this limitation also applies to the Log-Linear and Equal Area rules.

CONCLUSION

This paper has explained how the locations of velocity measurement points are calculated for the Equal Area, Log-Linear and Log-Tchebycheff rules, and in doing so has also revealed the corresponding assumed velocity profiles. Furthermore, we can now provide some tentative responses to the series of questions posed in the Introduction:

1. The Equal Area rule should be no less accurate for a circular duct than it is for a rectangular duct, even though it is not prescribed for circular ducts in ASHRAE 111 (ASHRAE 2017).
2. An unweighted log-linear rule with an arbitrary number of measurement points could be easily derived for a rectangular duct following the approach presented above. Whether it is worthwhile is another question, given the effectiveness of the Log-Tchebycheff rule in rectangular ducts.
3. The Log-Linear measurement points can be calculated to machine precision using the formulas derived in this paper. The precision of the Log-Tchebycheff points is limited to the precision of the Chebyshev points in Table 1. These could be calculated with greater precision following any of the methods described by Gautschi (1976).

Although Questions 4 and 5 are not resolved in this paper, a good starting point is provided for further research.

NOMENCLATURE

a = Intermediate variable
b = Intermediate variable
c = Intercept of a straight-line function
f = Indicates a function
g = Indicates a function
h = Slope of a straight line
i = Index for a series of points
j = Number of points per duct radius/half-width
k = Number of Chebyshev points
m = Interval or subregion index
n = Number of intervals or subregions
p = Degree of a polynomial equation
r = Radial coordinate
s = Distance coordinate
u = Longitudinal velocity
x = Distance across rectangular duct
y = Distance from wall of round duct
A = Cross-sectional area
C = Arbitrary coefficient
D = Diameter of round duct
E = Error term
R = Radius of round duct
S = Half-width of rectangular duct
W = Duct width
X = Non-dimensional distance across rectangular duct
ϵ = A very small number, e.g., 10^{-6}
ρ = Nondimensional radius

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